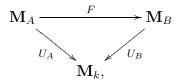# A NOTE ON CORING EXTENSIONS

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ABSTRACT. A notion of a coring extension is defined and it is related to the existence of an additive functor between comodule categories that factorises through forgetful functors. This correspondence between coring extensions and factorisable functors is illustrated by functors between categories of descent data. A category in which objects are corings and morphisms are coring extensions is also introduced.

1. INTRODUCTION

Given two algebras A, B over a commutative ring k, an algebra extension or an algebra map $B \to A$ can be equivalently characterised as a k-additive functor $F : \mathbf{M}_A \to \mathbf{M}_B$ with the factorisation property

$$\mathbf{M}_A \xrightarrow{\;\;F\;\;} \mathbf{M}_B$$
$$\begin{array}{ccc} & \searrow^{U_A} \quad \swarrow_{U_B} & \\ & \mathbf{M}_k, & \end{array}$$

where U_A, U_B are forgetful functors (cf. [8]). Through this correspondence, morphisms of k-algebras can be defined as functors having such a factorisation property. This point of view is taken up in a recent paper by Pareigis [9], in which functors between categories of entwined modules are studied, conditions for the factorisation property are derived and these are then suggested as the definition of morphisms between entwining structures. The resulting notion of morphisms of entwining structures is different from the one introduced earlier in [2]. Since any entwining structure gives rise to a coring such that the entwined modules can be identified with its comodules (cf. [3]), it is natural to look at the results of [9] from the coring point of view. This is the aim of the present note in which, rather than changing the established notion of a morphism of corings (cf. [7], [4, Section 24]), we introduce the notion of an *extension of corings* or a *coring extension* and show that such extensions arise from and – provided they satisfy a suitable purity condition (for example in the case of corings associated to entwining structures) – give rise to k-additive functors with a (suitable) factorisation property.

We work over a commutative associative ring k with a unit. All algebras are over k, associative and with a unit. The symbol $\otimes$ between k-modules and k-module maps means tensor product over k. As a rule we do not decorate $\otimes$ between elements, unless there is a danger of confusion. For a k-algebra A, the category of right A-modules

and right A-linear maps is denoted by $\mathbf{M}_A$. The product map in A is denoted by $\mu_A : A \otimes A \to A$ and the unit (either as an element of A or as a k-linear map $k \to A$) is denoted by 1_A. Given a k-algebra A, coproduct in an A-coring $\mathcal{C}$ is denoted by $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, and the counit is denoted by $\varepsilon_{\mathcal{C}} : \mathcal{C} \to A$. We use the Sweedler sigma notation, i.e., for all $c \in \mathcal{C}$,

$$\Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes c_{(2)}, \qquad (\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}}(c) = (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

etc. For an A-coring $\mathcal{C}$, the *left dual ring* is defined as a k-module ${}^*\mathcal{C} = \mathrm{Hom}_{A-}(\mathcal{C}, A)$ with the unit $\varepsilon_{\mathcal{C}}$ and the product, for all $f, g \in {}^*\mathcal{C}$, $c \in \mathcal{C}$, $f * g(c) = \sum g(c_{(1)} f(c_{(2)}))$.

The category of right $\mathcal{C}$-comodules and right $\mathcal{C}$-colinear maps is denoted by $\mathbf{M}^{\mathcal{C}}$. For a right $\mathcal{C}$-comodule M, $\varrho^M : M \to M \otimes_A \mathcal{C}$ denotes a coaction. Recall that $\mathbf{M}^{\mathcal{C}}$ is built upon the category of right A-modules, in the sense that every right $\mathcal{C}$-comodule is a right A-module, coactions and morphisms are right A-linear maps (with additional compatibility conditions). On elements, ϱ^M is denoted by the Sweedler notation $\varrho^M(m) = \sum m_{(0)} \otimes m_{(1)}$ (but see an exception in the proof of Theorem 2.6). Similar notational conventions apply to coalgebras and their comodules. A detailed account of the theory of corings and comodules can be found in [4].
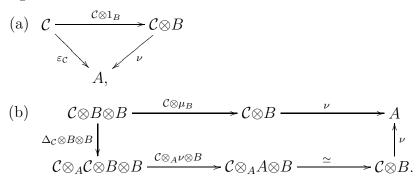
2. Extensions of corings and factorisable functors

Recall that, given an A-coring $\mathcal{C}$ and a B-coring $\mathcal{D}$, a $(\mathcal{C}, \mathcal{D})$-bicomodule is a left $\mathcal{C}$-comodule that is at the same time a right $\mathcal{D}$-comodule with $\mathcal{C}$-colinear $\mathcal{D}$-coaction. The $\mathcal{C}$-colinearity of $\mathcal{D}$-coaction is equivalent to $\mathcal{D}$-colinearity of $\mathcal{C}$-coaction.

Definition 2.1. Let A and B be k-algebras. A B-coring $\mathcal{D}$ is called a *right extension* of an A-coring $\mathcal{C}$ provided $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{D})$-bicomodule with the left regular coaction $\Delta_{\mathcal{C}}$.

For example, if $\mathcal{C}$ and $\mathcal{D}$ are A-corings and $\gamma : \mathcal{C} \to \mathcal{D}$ is an A-coring morphism, then $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{D})$-bicomodule with the left regular coaction $\Delta_{\mathcal{C}}$ and the right coaction $\varrho^{\mathcal{C}} = (\mathcal{C} \otimes_A \gamma) \circ \Delta_{\mathcal{C}}$. Thus any A-coring morphism gives rise to a coring extension.

Definition 2.1 implies in particular that if $\mathcal{D}$ is a right extension of $\mathcal{C}$, then necessarily $\mathcal{C}$ is a right B-module and $\Delta_{\mathcal{C}}$ is a right B-linear map. This leads to the following

Definition 2.2. Let A and B be k-algebras. An A-coring $\mathcal{C}$ is said to *measure B to A* if there exists a left A-linear map $\nu : \mathcal{C} \otimes B \to A$ rendering commutative the following diagrams:

(a)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{C} \otimes 1_B} & \mathcal{C} \otimes B \\ & {}_{\varepsilon_{\mathcal{C}}} \searrow & \downarrow {}^{\nu} \\ & A, & \end{array}$$

(b)
$$\begin{array}{ccccc} \mathcal{C} \otimes B \otimes B & \xrightarrow{\mathcal{C} \otimes \mu_B} & \mathcal{C} \otimes B & \xrightarrow{\nu} & A \\ {}_{\Delta_{\mathcal{C}} \otimes B \otimes B} \downarrow & & & & \uparrow {}^{\nu} \\ \mathcal{C} \otimes_A \mathcal{C} \otimes B \otimes B & \xrightarrow{\mathcal{C} \otimes_A \nu \otimes B} & \mathcal{C} \otimes_A A \otimes B & \xrightarrow{\simeq} & \mathcal{C} \otimes B. \end{array}$$

The map ν is called a *$\mathcal{C}$-measuring* of B to A.

Proposition 2.3. *Let $\mathcal{C}$ be an A-coring and B an algebra. $\mathcal{C}$-measurings of B to A are in bijective correspondence with algebra maps $B \to {}^*\mathcal{C}$.*

Proof. There is a bijective correspondence between left A-module maps $\nu : \mathcal{C} \otimes B \to A$ and k-linear maps $\chi : B \to {}^*\mathcal{C}$ provided by the hom-tensor isomorphism

$$\mathrm{Hom}_{A-}(\mathcal{C} \otimes B, A) \simeq \mathrm{Hom}_k(B, \mathrm{Hom}_{A-}(\mathcal{C}, A)) = \mathrm{Hom}_k(B, {}^*\mathcal{C}).$$

Explicitly, for all $b \in B$, $c \in \mathcal{C}$, $\nu(c \otimes b) = \chi(b)(c)$. Since the counit $\varepsilon_\mathcal{C}$ of $\mathcal{C}$ is the unit in ${}^*\mathcal{C}$, the map χ is unital if and only if $\nu(c \otimes 1_B) = \chi(1_B)(c) = \varepsilon_\mathcal{C}(c)$. Thus the unitality of χ is equivalent to the commutativity of the diagram (a) in Definition 2.2 for ν. Second, the multiplicativity of χ means that, for all $b, b' \in B$, $\chi(bb') = \chi(b) * \chi(b')$, i.e., for all $c \in \mathcal{C}$,

$$\chi(bb')(c) = \sum \chi(b')(c_{(1)} \chi(b)(c_{(2)})).$$

Therefore, χ is a multiplicative map if and only if

$$\begin{aligned}
\nu(c \otimes bb') &= \chi(bb')(c) = \sum \chi(b')(c_{(1)} \chi(b)(c_{(2)})) \\
&= \sum \nu(c_{(1)} \chi(b)(c_{(2)}) \otimes b') = \sum \nu(c_{(1)} \nu(c_{(2)} \otimes b) \otimes b'),
\end{aligned}$$

i.e., the diagram (b) in Definition 2.2 is commutative. $\square$

In view of Proposition 2.3, any $B^{op}|A$-*coring* in the sense of [11, Definition 3.5], any *right rational pairing* of corings in the sense of [6] or a *measuring left A-pairing* of [1] are examples of a $\mathcal{C}$-measuring. We illustrate the notion of a $\mathcal{C}$-measuring with a number of additional examples.

Examples 2.4.

(1) An algebra A, viewed as a trivial A-coring, measures B to A if and only if there is an algebra map $B \to A$.

(2) Given algebras A and B, let Σ be a (B, A)-bimodule that is finitely generated and projective as a right A-module and let $\mathcal{C} = \Sigma^* \otimes_B \Sigma$ be the corresponding comatrix A-coring (cf. [5]). Then $\mathcal{C}$-measurings of B to A are in bijective correspondence with right B-module structures on Σ that make Σ a (B, B)-bimodule.

(3) Given an algebra map $\iota : B \to A$, take $\mathcal{C} = A \otimes_B A$ the canonical Sweedler coring. Fix a left B-module structure on A provided by the map ι, i.e., $ba := \iota(b)a$. Then $\mathcal{C}$-measurings of B to A are in bijective correspondence with right B-module structures on A that make A a (B, B)-bimodule.

(4) Let A be a k-algebra and C be a k-coalgebra with coproduct Δ_C and counit ε_C. If $\mathcal{C} = A \otimes C$ is the coring associated to an entwining structure (A, C, ψ) then $\mathcal{C}$-measurings of B to A are in bijective correspondence with *entwined measurings* in the sense of [9, Remark 2.2], i.e., with k-linear maps $f : C \otimes B \to A$ making the following diagrams:

$$\begin{array}{ccc}
C & \xrightarrow{\;\;C \otimes 1_B\;\;} & C \otimes B \\
{\scriptstyle \varepsilon_C} \downarrow & & \downarrow {\scriptstyle f} \\
k & \xrightarrow[\;\;1_A\;\;]{} & A
\end{array}$$

and

$$C{\otimes}B{\otimes}B \xrightarrow{\ C{\otimes}\mu_B\ } C{\otimes}B \xrightarrow{\ f\ } A$$

$$\Delta_C{\otimes}B{\otimes}B \downarrow \qquad\qquad\qquad\qquad\qquad \uparrow \mu_A$$

$$C{\otimes}C{\otimes}B{\otimes}B \xrightarrow{\ C{\otimes}f{\otimes}B\ } C{\otimes}A{\otimes}B \xrightarrow{\ \psi{\otimes}B\ } A{\otimes}C{\otimes}B \xrightarrow{\ A{\otimes}f\ } A{\otimes}A$$

commute.

Check.(1) This follows immediately from Proposition 2.3, since $^*A \simeq A$ as k-algebras.

(2) Recall from [5] that $\mathcal{C} = \Sigma^*{\otimes}_B\Sigma$ is an A-coring with coproduct and counit, for all $s \in \Sigma$, $s^* \in \Sigma^*$,

$$\Delta_{\mathcal{C}}(s^*{\otimes}_Bs) = \sum_{i \in I} s^*{\otimes}_Be_i{\otimes}_Ae_i^*{\otimes}_Bs, \qquad \varepsilon_{\mathcal{C}}(s^*{\otimes}_Bs) = s^*(s),$$

where $\{e_i \in \Sigma, e_i^* \in \Sigma^*\}_{i \in I}$ is a finite dual basis of Σ_A. Recall also that $^*\mathcal{C} \simeq \mathrm{End}_{B-}(\Sigma)^{op}$, where the product in $\mathrm{End}_{B-}(\Sigma)$ is given by $fg(s) = f(g(s))$. Therefore, by Proposition 2.3, $\mathcal{C}$-measurings are in bijective correspondence with anti-algebra maps $B \to \mathrm{End}_{B-}(\Sigma)$, i.e., with right B-module structures on Σ such that Σ is a (B,B)-bimodule.

(3) This is a special case of (2), simply take $\Sigma = A$ and view A as a left B-module via the map ι.

(4) Recall that an entwining structure consists of an algebra $(A,\mu_A,1_A)$, a coalgebra $(C,\Delta_C,\varepsilon_C)$ and a k-linear map $\psi : C{\otimes}A \to A{\otimes}C$ satisfying a number of conditions (cf., e.g., [4, Section 32]). In this case, $\mathcal{C} = A{\otimes}C$ is an A-bimodule via $a(a'{\otimes}c)a'' = aa'\psi(c{\otimes}a'')$ and it has a coproduct and counit $\Delta_{\mathcal{C}}(a{\otimes}c) = a{\otimes}\Delta_C(c)$, $\varepsilon_{\mathcal{C}}(a{\otimes}c) = a\varepsilon_C(c)$. In the view of the isomorphism $\mathrm{Hom}_{A-}(A{\otimes}C{\otimes}B, A) \simeq \mathrm{Hom}_k(C{\otimes}B, A)$ any $\mathcal{C}$-measuring $\nu : \mathcal{C}{\otimes}B \to A$ corresponds to a k-linear map $f : C{\otimes}B \to A$, via $\nu(a{\otimes}c{\otimes}b) = af(c{\otimes}b)$. With the help of this identification one immediately checks that the diagram (a) in Definition 2.2 for ν is equivalent to the first of diagrams in (3) for f. As to the second pair of diagrams, introduce the explicit notation $\psi(c{\otimes}a) = \sum_\alpha a_\alpha{\otimes}c^\alpha$, take any $c \in C$, $b, b' \in B$, use the diagram (b) in Definition 2.2 and the definition of the right A-multiplication on $\mathcal{C}$ to compute

$$\begin{aligned}
f(c{\otimes}bb') &= \nu(1_A{\otimes}c{\otimes}bb') = \sum \nu((1_A{\otimes}c_{(1)})\nu(1_A{\otimes}c_{(2)}{\otimes}b){\otimes}b') \\
&= \sum_\alpha \nu(f(c_{(2)}{\otimes}b)_\alpha{\otimes}c_{(1)}{}^\alpha{\otimes}b') = \sum_\alpha f(c_{(2)}{\otimes}b)_\alpha f(c_{(1)}{}^\alpha{\otimes}b').
\end{aligned}$$

This is exactly the contents of the second of the diagrams in (3). Similarly one proves that if f makes this diagram commutative, then also ν renders commutative the diagram (b) in Definition 2.2. $\square$

In particular, if in Example 2.4 (4) the trivial entwining $\psi : C{\otimes}A \to A{\otimes}C$, $c{\otimes}a \mapsto a{\otimes}c$ is taken, then $\mathcal{C} = A{\otimes}C$ measures B to A if and only if C measures B to A in the sense of Sweedler [10, p. 138] (this justifies the choice of the name). Also, the combination of Example 2.4 (4) and Proposition 2.3 leads to an equivalent description of entwined measurings as algebra maps $B \to \#_\psi(C, A)$, where $\#_\psi(C, A)$ is a ψ-twisted

convolution algebra defined as a k-module $\mathrm{Hom}_k(C, A)$ with the unit ε_C and with the product, for all $f, g \in \mathrm{Hom}_k(C, A)$ and $c \in C$,

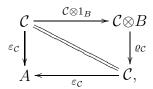$$(f \#_\psi g)(c) = \sum_\alpha f(c_{(2)})_\alpha g(c_{(1)}{}^\alpha).$$

This follows immediately from the fact that $\#_\psi(C, A)$ is isomorphic to the left dual ring of the coring $\mathcal{C} = A \otimes C$ associated to an entwining structure (A, C, ψ).

Examples 2.4 indicate that the notion of a coring measuring can be understood as one that unifies the notions of an algebra map, a bimodule structure and entwined measuring. The relationship between a measuring and a coring extension is revealed in the following

Lemma 2.5. *Let A, B be k-algebras and let $\mathcal{C}$ be an A-coring. Then the following statements are equivalent:*

(1) there exists a right B-module structure on $\mathcal{C}$ such that $\mathcal{C}$ is an (A, B)-bimodule and the coproduct $\Delta_{\mathcal{C}}$ is a B-linear map;

(2) $\mathcal{C}$ measures B to A.

Proof. $(1) \Rightarrow (2)$ Suppose that $\mathcal{C}$ is an (A, B)-bimodule with a B-linear coproduct $\Delta_{\mathcal{C}}$, and let $\varrho_{\mathcal{C}} : \mathcal{C} \otimes B \to \mathcal{C}$ be the right B-multiplication. Define $\nu = \varepsilon_{\mathcal{C}} \circ \varrho_{\mathcal{C}} : \mathcal{C} \otimes B \to A$. The condition (a) in Definition 2.2 for ν follows then from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathcal{C} \otimes 1_B} & \mathcal{C} \otimes B \\
{\scriptstyle \varepsilon_{\mathcal{C}}} \downarrow & & \downarrow {\scriptstyle \varrho_{\mathcal{C}}} \\
A & \xleftarrow{\varepsilon_{\mathcal{C}}} & \mathcal{C},
\end{array}
$$

in which the right upper triangle is commutative by the unitality of the multiplication $\varrho_{\mathcal{C}}$. Since $\Delta_{\mathcal{C}}$ is a right B-module morphism,

$$
\begin{array}{ccccc}
\mathcal{C} \otimes B & \xrightarrow{\varrho_{\mathcal{C}}} & \mathcal{C} & & \\
{\scriptstyle \Delta_{\mathcal{C}} \otimes B} \downarrow & & \downarrow {\scriptstyle \Delta_{\mathcal{C}}} & \searrow {\scriptstyle \simeq} & \\
\mathcal{C} \otimes_A \mathcal{C} \otimes B & \xrightarrow{\mathcal{C} \otimes_A \varrho_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}} & \mathcal{C} \otimes_A A
\end{array}
$$

is a commutative diagram. The right-hand triangle is simply the counit axiom. Note that the composition of maps in the bottow row equals $\mathcal{C} \otimes_A \nu$. Since $\varrho_{\mathcal{C}}$ is an associative multiplication and the above diagram commutes, we obtain the following commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{C} \otimes_A \mathcal{C} \otimes B \otimes B & \xleftarrow{\Delta_{\mathcal{C}} \otimes B \otimes B} & \mathcal{C} \otimes B \otimes B & \xrightarrow{\mathcal{C} \otimes \mu_B} & \mathcal{C} \otimes B & \xrightarrow{\varrho_{\mathcal{C}}} & \mathcal{C} \\
{\scriptstyle \mathcal{C} \otimes_A \nu \otimes B} \downarrow & & \downarrow {\scriptstyle \varrho_{\mathcal{C}} \otimes B} & & & & \downarrow {\scriptstyle \varepsilon_{\mathcal{C}}} \\
\mathcal{C} \otimes_A A \otimes B & \xrightarrow{\simeq} & \mathcal{C} \otimes B & \xrightarrow{\varrho_{\mathcal{C}}} & \mathcal{C} & \xrightarrow{\varepsilon_{\mathcal{C}}} & A.
\end{array}
$$

The outer rectangle in the above diagram is equivalent to the condition (b) in Definition 2.2 for $\nu = \varepsilon_{\mathcal{C}} \circ \varrho_{\mathcal{C}}$. Thus we conclude that ν is a $\mathcal{C}$-measuring of B to A as required.

(2) $\Rightarrow$ (1) Given a $\mathcal{C}$-measuring ν, define $\varrho_{\mathcal{C}} : \mathcal{C} \otimes B \to \mathcal{C}$ by $c \otimes b \mapsto \sum c_{(1)} \nu(c_{(2)} \otimes b)$. Then condition (a) in Definition 2.2 for ν implies that, for all $c \in \mathcal{C}$,

$$\varrho_{\mathcal{C}}(c \otimes 1_B) = \sum c_{(1)} \nu(c_{(2)} \otimes 1_B) = \sum c_{(1)} \varepsilon_{\mathcal{C}}(c_{(2)}) = c.$$

Furthermore, the use of condition (b) (in the second equality below), the definition of $\varrho_{\mathcal{C}}$ in terms of ν, and the right A-linearity of $\Delta_{\mathcal{C}}$ give, for all $c \in \mathcal{C}$, $b, b' \in B$,

$$\begin{aligned} \varrho_{\mathcal{C}}(c \otimes bb') &= \sum c_{(1)} \nu(c_{(2)} \otimes bb') = \sum c_{(1)} \nu(c_{(2)} \nu(c_{(3)} \otimes b) \otimes b') \\ &= \sum \varrho_{\mathcal{C}}(c_{(1)} \nu(c_{(2)} \otimes b) \otimes b') = \varrho_{\mathcal{C}}(\varrho_{\mathcal{C}}(c \otimes b) \otimes b'). \end{aligned}$$

Thus $\mathcal{C}$ is a right B-module with the multiplication $\varrho_{\mathcal{C}}$. The map $\varrho_{\mathcal{C}}$ is a composition of left A-linear maps, hence a left A-linear map, i.e., $\mathcal{C}$ is an (A, B)-bimodule. Finally, since the coproduct is a coassociative right A-linear map, for all $b \in B$ and $c \in \mathcal{C}$,
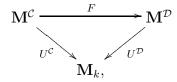
$$\Delta_{\mathcal{C}}(\varrho_{\mathcal{C}}(c \otimes b)) = \sum \Delta_{\mathcal{C}}(c_{(1)} \nu(c_{(2)} \otimes b)) = \sum c_{(1)} \otimes c_{(2)} \nu(c_{(3)} \otimes b) = \sum c_{(1)} \otimes \varrho_{\mathcal{C}}(c_{(2)} \otimes b).$$

This means that the coproduct $\Delta_{\mathcal{C}}$ is a right B-linear map as required. $\square$

The main result of this note is contained in the following

Theorem 2.6. *Let $\mathcal{C}$ be an A-coring and $\mathcal{D}$ be a B-coring.*

(1) *If there exists a k-additive functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{\mathcal{D}}$ with a factorisation property*

$$\mathbf{M}^{\mathcal{C}} \xrightarrow{\;\;F\;\;} \mathbf{M}^{\mathcal{D}}$$
$$U^{\mathcal{C}} \searrow \qquad \swarrow U^{\mathcal{D}}$$
$$\mathbf{M}_k,$$

where $U^{\mathcal{C}}$, $U^{\mathcal{D}}$ are forgetful functors, then $\mathcal{D}$ is a right extension of $\mathcal{C}$.

(2) *Let $\mathcal{D}$ be a right extension of $\mathcal{C}$ such that, for all $\mathcal{C}$-comodules (N, ϱ^N), the right B-module map $\varrho^N \otimes_A \mathcal{C} - N \otimes_A \Delta_{\mathcal{C}}$ is $\mathcal{D} \otimes_B \mathcal{D}$-pure. Then there exists a k-additive functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{\mathcal{D}}$ with a factorisation property as in (1).*

Proof. (1) Let $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{\mathcal{D}}$ be a k-additive functor that factorises through forgetful functors $U^{\mathcal{C}}$ and $U^{\mathcal{D}}$. The factorisation property means that for any $M \in \mathbf{M}^{\mathcal{C}}$, $F(M) = M$ as k-modules. Similarly, for any morphism f in $\mathbf{M}^{\mathcal{C}}$, $F(f) = f$ as k-linear maps. This implies that for any right $\mathcal{C}$-comodule M there exist an action $\varrho_M : M \otimes B \to M$ and a coaction $\varrho^M : M \to M \otimes_B \mathcal{D}$, and any k-linear map f that is a morphism in $\mathbf{M}^{\mathcal{C}}$ is also a morphism in $\mathbf{M}^{\mathcal{D}}$ (the functoriality of action and coaction). In particular, $\mathcal{C}$ is a right $\mathcal{C}$-comodule with the regular coaction $\Delta_{\mathcal{C}}$, hence $\mathcal{C}$ is a right B-module and there exists a right $\mathcal{D}$-coaction on $\mathcal{C}$, $\varrho^{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_B \mathcal{D}$. In addition, $\Delta_{\mathcal{C}}$ is a left A-module map. Equivalently, for any $a \in A$, the right $\mathcal{C}$-colinear map $\ell_a : \mathcal{C} \to \mathcal{C}$, $c \mapsto ac$ is a morphism in $\mathbf{M}^{\mathcal{C}}$. Thus the k-linear map ℓ_a is a morphism in $\mathbf{M}^{\mathcal{D}}$, i.e., $\varrho^{\mathcal{C}}$ is a left A-module map. Furthermore, $\mathcal{C} \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-comodule with the coaction $\mathcal{C} \otimes_A \Delta_{\mathcal{C}}$, hence it is a right B-module and there exists a right $\mathcal{D}$-coaction, $\varrho^{\mathcal{C} \otimes_A \mathcal{C}} : \mathcal{C} \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C} \otimes_B \mathcal{D}$. For any $c \in \mathcal{C}$, consider a right $\mathcal{C}$-comodule map $\ell^c : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, $c' \mapsto c \otimes c'$. Since $\Delta_{\mathcal{C}}$ is a right $\mathcal{C}$-comodule map too, the functoriality of $\mathcal{D}$-coactions implies that, for all $c \in \mathcal{C}$,

$$(\ell^c \otimes_B \mathcal{D}) \circ \varrho^{\mathcal{C}} = \varrho^{\mathcal{C} \otimes_A \mathcal{C}} \circ \ell^c, \qquad \varrho^{\mathcal{C} \otimes_A \mathcal{C}} \circ \Delta_{\mathcal{C}} = (\Delta_{\mathcal{C}} \otimes_B \mathcal{D}) \circ \varrho^{\mathcal{C}},$$

in $\mathbf{M}_k$. Putting these two together we obtain,

$$
\begin{aligned}
\sum c_{(1)} \otimes \varrho^{\mathcal{C}}(c_{(2)}) &= \sum (\ell^{c_{(1)}} \otimes_B \mathcal{D}) \circ \varrho^{\mathcal{C}}(c_{(2)}) = \sum \varrho^{\mathcal{C} \otimes_A \mathcal{C}} \circ \ell^{c_{(1)}}(c_{(2)}) \\
&= \sum \varrho^{\mathcal{C} \otimes_A \mathcal{C}}(c_{(1)} \otimes c_{(2)}) = \varrho^{\mathcal{C} \otimes_A \mathcal{C}} \circ \Delta_{\mathcal{C}}(c) \\
&= (\Delta_{\mathcal{C}} \otimes_B \mathcal{D}) \circ \varrho^{\mathcal{C}}(c).
\end{aligned}
$$

This means that the coaction $\varrho^{\mathcal{C}}$ is a left $\mathcal{C}$-comodule map, hence $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule, i.e., $\mathcal{D}$ is a coring extension of $\mathcal{C}$.

(2) This is contained in [4, 22.3, Erratum]. In detail, suppose that $\mathcal{D}$ is a coring extension of $\mathcal{C}$ and let $\nu : \mathcal{C} \otimes B \to A$ be the measuring corresponding to the right B-multiplication on $\mathcal{C}$ as in Lemma 2.5. Write $\sigma : \mathcal{C} \to \mathcal{C} \otimes_B \mathcal{D}$ for the right $\mathcal{D}$-coaction on $\mathcal{C}$. Define a k-linear functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{\mathcal{D}}$ as follows.

Take any right $\mathcal{C}$-comodule M and define a map $\varrho_M : M \otimes B \to M$, $m \otimes b \mapsto \sum m_{(0)} \nu(m_{(1)} \otimes b)$. Following the same steps as in the proof $(2) \Rightarrow (1)$ of Lemma 2.5 one easily verifies that M is a right B-module with multiplication ϱ_M. We write $m.b := \varrho_M(m \otimes b)$. Furthermore, if $f : M \to N$ is a morphism in $\mathbf{M}^{\mathcal{C}}$, then for all $m \in M$, $b \in B$,

$$
\begin{aligned}
f(m.b) &= \sum f(m_{(0)} \nu(m_{(1)} \otimes b)) = \sum f(m_{(0)}) \nu(m_{(1)} \otimes b) \\
&= \sum f(m)_{(0)} \nu(f(m)_{(1)} \otimes b) = f(m).b.
\end{aligned}
$$

The second equality follows from the A-linearity of f, while the third one is a consequence of the fact that f is a $\mathcal{C}$-comodule map. The first and last equalities follow from the definition of the B-multiplication ϱ_M on M. Therefore, f is a right B-linear map, and thus we have constructed a functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$. Now we need to define a $\mathcal{D}$-coaction on any $\mathcal{C}$-comodule.

Start with the right $\mathcal{C}$-comodule isomorphism $M \simeq M \square_{\mathcal{C}} \mathcal{C}$ (cf. [4, 22.4]) provided by the right $\mathcal{C}$-coaction on M. Here $\square_{\mathcal{C}}$ denotes the cotensor product over $\mathcal{C}$. By applying the above functor $\mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ we obtain a right B-module isomorphism and we can construct a map

$$
\begin{aligned}
\varrho^M : \ M \ &\xrightarrow{\ \simeq\ } M \square_{\mathcal{C}} \mathcal{C} \xrightarrow{\ M \square_{\mathcal{C}} \sigma\ } M \square_{\mathcal{C}} (\mathcal{C} \otimes_B \mathcal{D}) \simeq M \otimes_B \mathcal{D} \\
m \ &\longmapsto \hspace{5.5cm} \sum m_{(0)} \varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}) \otimes m_{(1)}{}^{[1]},
\end{aligned}
$$

where $\sum m_{(0)} \otimes m_{(1)} \in M \otimes_A \mathcal{C}$ denotes the $\mathcal{C}$-coaction on M, while $\sigma(c) = \sum c^{[0]} \otimes c^{[1]} \in \mathcal{C} \otimes_B \mathcal{D}$ denotes the $\mathcal{D}$-coaction on $\mathcal{C}$. Note that ϱ^M is well-defined by the purity assumption. We claim that ϱ^M is a right $\mathcal{D}$-coaction. First, ϱ^M is a right B-module map as a composition of B-module maps. Note that, for all $m \in M$,

$$
\sum m_{(0)} \varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}) \varepsilon_{\mathcal{D}}(m_{(1)}{}^{[1]}) = \sum m_{(0)} \varepsilon_{\mathcal{C}}(m_{(1)}) = m,
$$

so that ϱ^M is a counital map. It remains to check the coassociativity of ϱ^M. This is done by a rather lengthy but straightforward calculation, the details of which are

displayed below. Take any $m \in M$ and compute

$$(\varrho^M \otimes_B \mathcal{D}) \circ \varrho^M(m) = \sum \varrho^M(m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]})) \otimes m_{(1)}{}^{[1]}$$

$$= \sum m_{(0)(0)}\varepsilon_{\mathcal{C}}((m_{(0)(1)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}))^{[0]}) \otimes (m_{(0)(1)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}))^{[1]} \otimes m_{(1)}{}^{[1]}$$

$$= \sum m_{(0)}\varepsilon_{\mathcal{C}}((m_{(1)(1)}\varepsilon_{\mathcal{C}}(m_{(1)(2)}{}^{[0]}))^{[0]}) \otimes (m_{(1)(1)}\varepsilon_{\mathcal{C}}(m_{(1)(2)}{}^{[0]}))^{[1]} \otimes m_{(1)(2)}{}^{[1]}$$

$$= \sum m_{(0)}\varepsilon_{\mathcal{C}}((m_{(1)}{}^{[0]}{}_{(1)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}{}_{(2)}))^{[0]}) \otimes (m_{(1)}{}^{[0]}{}_{(1)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}{}_{(2)}))^{[1]} \otimes m_{(1)}{}^{[1]}$$

$$= \sum m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0][0]}) \otimes m_{(1)}{}^{[0][1]} \otimes m_{(1)}{}^{[1]}$$

$$= \sum m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}) \otimes m_{(1)}{}^{[1]}{}_{(1)} \otimes m_{(1)}{}^{[1]}{}_{(2)}$$

$$= (\mathcal{C} \otimes_B \Delta_{\mathcal{D}}) \circ \varrho^M(m).$$

The fourth equality is a consequence of the fact that $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{D})$-bicomodule and the purity assumption is used to derive the penultimate equality. Thus ϱ^M is a right $\mathcal{D}$-coaction.

We already know that if $f : M \to N$ is a morphism in $\mathbf{M}^{\mathcal{C}}$, then f is a right B-module map. Take any $m \in M$ and compute
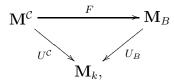
$$\varrho^N(f(m)) = \sum f(m)_{(0)}\varepsilon_{\mathcal{C}}(f(m)_{(1)}{}^{[0]}) \otimes f(m)_{(1)}{}^{[1]} = \sum f(m_{(0)})\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]}) \otimes m_{(1)}{}^{[1]}$$

$$= \sum f(m_{(0)}\varepsilon_{\mathcal{C}}(m_{(1)}{}^{[0]})) \otimes m_{(1)}{}^{[1]} = (f \otimes_B \mathcal{D}) \circ \varrho^M(m),$$

where the second equality follows from the $\mathcal{C}$-colinearity and the third one from the A-linearity of f. Thus f is a morphism of right $\mathcal{D}$-comodules. Put together, all this means that we have constructed a k-additive functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}^{\mathcal{D}}$. It is obvious from this construction that F factorises through the forgetful functors $U^{\mathcal{C}}$ and $U^{\mathcal{D}}$ as required. $\square$

In view of Lemma 2.5, a trivial B-coring B is a right extension of $\mathcal{C}$ if and only if $\mathcal{C}$ measures B to A. Thus Theorem 2.6 leads immediately to the following characterisation of measurings in terms of functors with a factorisation property.

Corollary 2.7. *Let A and B be k-algebras and let $\mathcal{C}$ be an A-coring. The following statements are equivalent:*
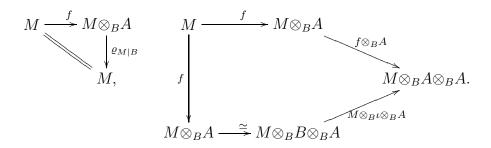
(1) $\mathcal{C}$ measures B to A;

(2) there exists a k-additive functor $F : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ with a factorisation property

$$\mathbf{M}^{\mathcal{C}} \xrightarrow{\;\;F\;\;} \mathbf{M}_B$$

$$U^{\mathcal{C}} \searrow \qquad \swarrow U_B$$

$$\mathbf{M}_k,$$

where $U^{\mathcal{C}}$, U_B are forgetful functors.

If $\mathcal{C}$ and $\mathcal{D}$ are corings corresponding to entwining structures (A, C, ψ), (B, D, Ψ) then Theorem 2.6 implies [9, Theorem 2.3]. As many general results about corings, Theorem 2.6 finds an application in non-commutative descent theory.
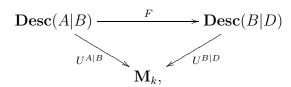
Given an algebra extension $\iota : B \to A$, the category of right comodules of the associated Sweedler coring $\mathcal{C} = A \otimes_B A$ is isomorphic to the category of *(right) descent data* $\mathbf{Desc}(A|B)$ (cf. [4, 25.4]). The objects in $\mathbf{Desc}(A|B)$ are pairs (M, f), where M is a right A-module and $f : M \to M \otimes_B A$ is a right A-module map rendering commutative the following diagrams

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & M \otimes_B A \\
& \searrow & \downarrow{\scriptstyle \varrho_{M|B}} \\
& & M,
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & M \otimes_B A \\
\downarrow{\scriptstyle f} & & \searrow{\scriptstyle f \otimes_B A} \\
& & M \otimes_B A \otimes_B A. \\
M \otimes_B A & \xrightarrow{\ \simeq\ } & M \otimes_B B \otimes_B A \\
& \nearrow{\scriptstyle M \otimes_B \iota \otimes_B A} &
\end{array}
$$

Here $\varrho_{M|B} : M \otimes_B A \to M$ is the factorised (through $M \otimes A \to M \otimes_B A$) A-multiplication on M. Using this identification of right comodules of the Sweedler coring $A \otimes_B A$ with (right) descent data of the ring extension $B \to A$ we can thus derive the following corollary of Theorem 2.6.

Corollary 2.8. *Let A, B, D be k-algebras and let $\iota_B : D \to B$ and $\iota_A : B \to A$ be algebra maps. Then the following statements are equivalent.*

(1) *There exists a k-additive functor $F : \mathbf{Desc}(A|B) \to \mathbf{Desc}(B|D)$ with a factorisation property*

$$
\begin{array}{ccc}
\mathbf{Desc}(A|B) & \xrightarrow{\ \ F\ \ } & \mathbf{Desc}(B|D) \\
& {\scriptstyle U^{A|B}} \searrow \quad \swarrow {\scriptstyle U^{B|D}} & \\
& \mathbf{M}_k, &
\end{array}
$$

where $U^{A|B}$, $U^{B|D}$ are forgetful functors.

(2) *View A as a left B-module via the map ι_A. There exist:*

(i) *a left B-linear right B-multiplication $\varrho_A : A \otimes B \to A$;*

(ii) *a (B, B)-bimodule map $\varphi : A \to A \otimes_B A \otimes_D B$ (A is a right B-module via ϱ_A) rendering commutative the following three diagrams:*

(a)
$$
\begin{array}{ccc}
A & \xrightarrow{\ \simeq\ } & B \otimes_B A \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \iota_A \otimes_B A} \\
A \otimes_B A \otimes_D B & \xrightarrow{\ A \otimes_B \varrho_{A|D}\ } & A \otimes_B A,
\end{array}
$$

(b)
$$
\begin{array}{ccccc}
A & \xrightarrow{\ \varphi\ } & A \otimes_B A \otimes_D B & \xrightarrow{\ A \otimes_B \varphi \otimes_D B\ } & A \otimes_B A \otimes_B A \otimes_D B \otimes_D B \\
\downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \mu_{A|B} \otimes_B A \otimes_D B \otimes_D B} \\
A \otimes_B A \otimes_D B & \xrightarrow{\ \simeq\ } & A \otimes_B A \otimes_D D \otimes_D B & \xrightarrow{\ A \otimes_B A \otimes_D \iota_B \otimes_D B\ } & A \otimes_B A \otimes_D B \otimes_D B,
\end{array}
$$

$$(c) \qquad A \xrightarrow{\simeq} B\otimes_B A \xrightarrow{\iota_A\otimes_B\varphi} A\otimes_B A\otimes_B A\otimes_D B$$

with vertical map φ on the left, and $A\otimes_B\iota_A\otimes_B A\otimes_D B$ on the right:

$$A\otimes_B A\otimes_D B \xrightarrow{\simeq} A\otimes_B B\otimes_B A\otimes_D B,$$

where $\varrho_{A|D} : A\otimes_D B \to A$ is the factorised (through $A\otimes B \to A\otimes_D B$) right B-multiplication ϱ_A and $\mu_{A|B} : A\otimes_B A \to A$ is the factorised (through $A\otimes A \to A\otimes_B A$) product μ_A in A.

Proof. The purity condition in Theorem 2.6 (2) always holds for Sweedler corings, hence part (1) is equivalent to the statement that $A\otimes_B A$ is a right coring extension of $B\otimes_D B$. The contents of this statement is contained in part (2). We only explain the origin of maps and diagrams in part (2), the details of the proof are left to the reader. By Example 2.4(3), there must exist a right B-multiplication ϱ_A as in (i). Furthermore, $A\otimes_B A$ must be a right $B\otimes_D B$-comodule, thus there exists a descent datum $(A\otimes_B A, f) \in \mathbf{Desc}(B|D)$. The map $f : A\otimes_B A \to A\otimes_B A\otimes_D B$ is a left A-module map as it corresponds to a coaction that is left $\mathcal{C}$-colinear. As a part of a descent datum, f is right B-linear. In view of the isomorphism $\mathrm{Hom}_{A,B}(A\otimes_B A, A\otimes_B A\otimes_D B) \simeq \mathrm{Hom}_{B,B}(A, A\otimes_B A\otimes_D B)$, f can be equivalently given as a (B,B)-bimodule map $\varphi : A \to A\otimes_B A\otimes_D B$, $a'\varphi(a) = f(a'\otimes a)$. The diagrams (a) and (b) are the defining diagrams for f as a part of a descent datum written in terms of φ. The diagram (c) expresses the fact that f is a left $A\otimes_B A$-comodule map, as it corresponds to a coaction that makes $A\otimes_B A$ into an $(A\otimes_B A, B\otimes_D B)$-comodule. $\square$

A functor obtained as a composition of any two functors between comodule categories that factorise through the forgetful functors also factorises through forgetful functors. The correspondence between such functors and coring extensions leads therefore to a category $\mathbf{CrgExt}_k^r$ in which objects are corings understood as pairs $(\mathcal{C}:A)$. Morphisms $(\mathcal{C}:A) \to (\mathcal{D}:B)$ are *pure coring extensions*, i.e. pairs $(\varrho_{\mathcal{C}}, \varrho^{\mathcal{C}})$ where $\varrho_{\mathcal{C}} : \mathcal{C}\otimes B \to \mathcal{C}$ is a left $\mathcal{C}$-colinear B-action and $\varrho^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}\otimes_B\mathcal{D}$ is a left $\mathcal{C}$-colinear $\mathcal{D}$-coaction, such that, for all $\mathcal{C}$-comodules (N, ϱ^N), the right B-module map $\varrho^N\otimes_A\mathcal{C} - N\otimes_A\Delta_{\mathcal{C}}$ is $\mathcal{D}\otimes_B\mathcal{D}$-pure. A composition of morphisms $(\mathcal{C}:A) \to (\mathcal{D}:B)$ and $(\mathcal{D}:B) \to (\mathcal{E}:D)$ is derived from the composition of corresponding functors and comes out as

$$\varrho_{\mathcal{D}} \bullet \varrho_{\mathcal{C}} : \mathcal{C}\otimes D \xrightarrow{\varrho^{\mathcal{C}}\otimes D} \mathcal{C}\otimes_B\mathcal{D}\otimes D \xrightarrow{\mathcal{C}\otimes_B\varrho_{\mathcal{D}}} \mathcal{C}\otimes_B\mathcal{D} \xrightarrow{\mathcal{C}\otimes_B\varepsilon_{\mathcal{D}}} \mathcal{C}\otimes_B B \simeq \mathcal{C}$$

and

$$\varrho^{\mathcal{D}} \bullet \varrho^{\mathcal{C}} : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}\square_{\mathcal{D}}\mathcal{D} \xrightarrow{\mathcal{C}\square_{\mathcal{D}}\varrho^{\mathcal{D}}} \mathcal{C}\square_{\mathcal{D}}(\mathcal{D}\otimes_D\mathcal{E}) \simeq \mathcal{C}\otimes_D\mathcal{E},$$

where the first isomorphism is provided by the coaction $\varrho^{\mathcal{C}|\mathcal{D}}$, while the second one is obtained with the help of the counit in $\mathcal{D}$ (compare the construction of coaction ϱ^M in the proof of Theorem 2.6 (2)).

Finally, we would like to point out that the results of this note can also be presented for left comodules of a coring thus leading to the notions of a left $\mathcal{C}$-measuring and a left coring extension. This is achieved by using the obvious left-right correspondence. Note, however, that a right coring extension is not necessarily a left coring extension, thus the left-right symmetry that exists for characterisation of algebra (or coalgebra) extensions does not exist in the general coring case.

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